A Group-Theoretical Method for Natanzon Potentials in Position-Dependent Mass Background by Means of Conformal Mappings

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Abstract A new manner for deriving the exact potentials is presented. By making use of conformal mappings, the general expression of the effective potentials deduced under the $\mathfrak{su}(1, 1)$ algebra can be brought back to the general Natanzon hypergeometric potentials.

1 Introduction

Exact solutions for some quantum mechanical systems endowed with position-dependent effective mass have attracted, in recent years, most attention on behalf of physicists [1–8]. Effective mass Schrödinger equation was introduced by BenDaniel and Duke [1] in order to explain the behavior of electrons in semiconductors. It has also many applications in the fields of material sciences and condensed matter physics [9–11].

Exact solvability of the Schrödinger equation was already discussed by employing various techniques [12–25]. The underlying thoughts behind these techniques might have different origins, while the applied technical approaches are rather similar. The group-theoretical approaches are one of these techniques and are useful for describing the bound-state problems involving dynamical groups [15–25]. Originally, these approaches were used to derive the Natanzon-class potentials [26] and their subclasses as, an example treated here, the Ginocchio potentials [28–31] using the new concept of the potential group [20–23] which connects all states that have the same energy but belong to different potential strengths.

In this paper, the conformal mappings [32–34] are presented as a means of generating the general Natanzon hypergeometric potentials (GNHP) under the $\mathcal{SU}(1, 1)$ group-representation. The central observation of the use of the conformal mappings consists in the fact that the variable z, as defined in the Natanzon-class potentials, varies in the interval

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[0, 1]; this latter will be regarded as the radius of the unit circle along the real-axis. This provides a systematic way for deriving useful conformal mappings of the domain D lying in the ξ-plan onto the interior of the unit circle $D_* = [0, 1]$ lying in the z-plan (see Appendix).

The plan of the present paper is as follow. In Sect. 2 we deduce the expression of the effective potential using the differential realization of the $\mathfrak{su}(1, 1)$ algebra, then exploiting the conformal mappings lead to the general Natanzon hypergeometric potentials endowed with the position-dependent mass. Section 3 deals with the generation of the Ginocchio potentials in its hyperbolic and polynomial forms. The final section will be devoted to discussions and an appendix was added, where the mathematical details about conformal mappings will be presented.

2 The $\mathfrak{su}(1, 1)$ Algebra and Natanzon-Class Potentials

The $\mathfrak{su}(1, 1)$ Lie algebra consists of the three generators $J_\pm$, J_0 satisfying the commutation relations [15–25]

$$[J_+, J_-] = -2J_\pm; \qquad [J_0, J_\pm] = \pm J_\pm, \tag{1}$$

where $J_\pm = J_\mp^+$. They are related to the Casimir invariant operator as

$$C = J_0^2 \mp J_0 - J_\pm J_\mp. \tag{2}$$

Note that the eigenstates of C and J_0, with eigenvalues $\langle C \rangle \equiv c = j(j+1)$ and $\langle J_0 \rangle = j_0$, serve as basis for the irreducible representation of $\mathcal{SU}(1, 1)$, and can be labelled $|j, j_0\rangle$. The allowed values of j_0 are related to j by [15, 16]

$$j_0 = n + \frac{1}{2} + \sqrt{c + \frac{1}{4}}, \tag{3}$$

where $n = 0, 1, 2, \ldots$. According to [24, 25] the generators $J_\pm$, J_0 can be expressed in terms of the first-derivative

$$J_\pm = e^{\pm i\varphi}[\pm h(x)\partial_x \pm g(x) + f(x)J_0 + k(x)], \quad J_0 = -i\partial_\varphi, \tag{4}$$

where we have used the abbreviation $\partial_\Sigma = \frac{d}{d\Sigma}$, with $\Sigma = x, \varphi$. Equations (1) and (4) provide the restrictions which determine the shape of the real functions $h(x), k(x)$ and $f(x)$ through the differential equations

$$f^2(x) - h(x)\partial_x f(x) = 0, \qquad h(x)\partial_x k(x) - f(x)k(x) = 0. \tag{5}$$

By applying a variable transformation $h(x) \to h(x)\frac{d\xi(x)}{dx} = \xi(x)$ on (5), we obtain the solutions of $f(x)$ and $k(x)$

$$f(x) = \frac{1 + a\xi^2(x)}{1 - a\xi^2(x)}, \qquad k(x) = \frac{\delta\xi(x)}{1 - a\xi^2(x)}, \tag{6}$$

where a and δ are constants of integration.

Inserting (6) into (2) and taking into account (4), we get

$$C = \frac{\xi^2}{\xi'^2}\partial_x^2 + \frac{\xi}{\xi'}\left[2g - \frac{\xi\xi''}{\xi'^2} - \frac{2\xi^2}{1 - \xi^2}\right]\partial_x + \frac{\xi}{\xi'}g' + g^2$$
$$- \frac{1 + \xi^2}{1 - \xi^2}g - \xi\frac{(\delta + 2j_0\xi)(2j_0 + \delta\xi)}{(1 - \xi^2)^2}, \tag{7}$$

where the prime denotes the derivative with respect to x. Equation (7) corresponds to the appropriate-parameter choice $a = 1$.

On the other hand, the general form of the Hamiltonian introduced by von Roos [2] for the spatially varying mass $M(x) = m_0 m(x)$, where $m(x)$ is a dimensionless mass, reads

$$H_{VR} = \frac{1}{4} \left[m^\eta(x) \mathbf{p} m^\varepsilon(x) \mathbf{p} m^\rho(x) + m^\rho(x) \mathbf{p} m^\varepsilon(x) \mathbf{p} m^\eta(x) \right] + V(x), \tag{8}$$

where $m_0 = 1$ and the restriction on the parameters η, ε and ρ checks the condition $\eta + \varepsilon + \rho = -1$. Here $\mathbf{p}(\equiv -i\hbar\partial_x)$ is the momentum. In the natural units ($\hbar = c = 1$), the Hamiltonian H_{VR} becomes

$$H_{VR} = -\frac{1}{2m}\partial_x^2 + \frac{m'}{2m^2}\partial_x + (1+\varepsilon)\frac{m''}{4m^2} - \left[\eta(\eta+\varepsilon+1)+\varepsilon+1\right]\frac{m'^2}{2m^3} + V(x). \tag{9}$$

By introducing the eigenfunctions [19]

$$\Psi_\lambda(x) = 2\lambda m(x)\frac{\xi^2(x)}{\xi'^2(x)}\psi(x), \tag{10}$$

where λ is real, the Hamiltonian (9) becomes

$$H_{VR} = -\lambda\frac{\xi^2}{\xi'^2}\partial_x^2 - \frac{\lambda\xi}{\xi'}\left[4 + \frac{m'\xi}{m\xi'} - \frac{4\xi\xi''}{\xi'^2}\right]\partial_x + \frac{2\lambda\xi}{\xi'^2}\left[3\xi'' + \frac{\xi\xi'''}{\xi'} - \frac{3\xi\xi''^2}{\xi'^2}\right]$$
$$+ \frac{\lambda m'\xi^2}{m\xi'^2}\left[\frac{2(\xi\xi''-\xi'^2)}{\xi\xi'} + \frac{(\varepsilon-1)m''}{2m'} - (1+\eta)(\eta+\varepsilon)\frac{m'}{m}\right]$$
$$- 2\lambda + 2\lambda m\frac{\xi^2}{\xi'^2}V(x). \tag{11}$$

The Schrödinger equation can be solved once equating it to the eigenvalues equation of the Casimir invariant operator of the $\mathfrak{su}(1,1)$ algebra following [17]

$$(H_{VR} - E)\Psi_\lambda(x) = \Lambda(x)(C - c)\Psi_\lambda(x) = 0, \tag{12}$$

where $\Lambda(x)$ is some real function to be determined. This requirement provides the identities

$$\Lambda(x) = -\lambda, \qquad g(x) = \frac{2-\xi^2(x)}{1-\xi^2(x)} - \frac{3\xi(x)\xi''(x)}{2\xi'^2(x)} + \frac{m'(x)\xi(x)}{2m(x)\xi'(x)}. \tag{13}$$

Inserting $g(x)$, $g'(x)$ and $g^2(x)$ as defined in (13) into (12), taking into consideration (7) and (11), we end up with

$$V_{eff}(x) - E = \frac{2\delta j_0 + \xi(\delta^2 + 4j_0^2 - 1 + 2\delta j_0\xi)}{2m\xi(1-\xi^2)^2}\xi'^2$$
$$+ \frac{c}{2m}\frac{\xi'^2}{\xi^2} + \frac{3}{8m}\frac{\xi''^2}{\xi'^2} - \frac{1}{4m}\frac{\xi'''}{\xi'} + V_m^{(\eta,\varepsilon)}(x), \tag{14}$$

where the potential $V_m^{(\eta,\varepsilon)}(x)$ depends only on the mass-term is expressed as

$$V_m^{(\eta,\varepsilon)}(x) = \left[(1+2\eta)^2 + 4\varepsilon(1+\eta)\right]\frac{m'^2(x)}{8m^3(x)} - \frac{\varepsilon}{4}\frac{m''(x)}{m^2(x)}. \tag{15}$$

Now to derive the GNHP and the different steps of calculations that can arise, we introduce first a transformation deduced from the conformal mappings (see Appendix)

$$\xi(x) = \frac{1 + i\sqrt{z(x)}}{1 - i\sqrt{z(x)}}, \tag{16}$$

followed by replacing $z(x) \rightarrow -z(x)$.[1] Here, we suppose that the variable function $z(x)$ varies in the interval $[0, 1]$. Then (14) becomes

$$E - V_{eff}(x) = \frac{pz - q - 1}{4z^2(1 - z)}\frac{z'^2}{2m} - \frac{c}{z(1 - z^2)}\frac{z'^2}{2m} - \frac{3}{8m}\frac{z''^2}{z'^2} + \frac{1}{4m}\frac{z'''}{z'} - V_m^{(\eta,\varepsilon)}(x), \tag{17}$$

with the specific-parameter choice of p and q are given by

$$p \equiv t - 1 = \left(\frac{\delta - 2j_0}{2}\right)^2 - 1, \qquad q + 1 \equiv r - 1 = \left(\frac{\delta + 2j_0}{2}\right)^2 - 1. \tag{18}$$

Without loss of generality, let us assume that the function $z(x)$ is related to a certain generating function, namely, $S(x)$ given by

$$S(x) = \frac{z'^2(x)}{2m(x)}. \tag{19}$$

By performing a formal derivative of (17) taking into account (19), we obtain

$$S(x) = -\frac{4z^2(x)}{\frac{4z(x)}{[1 - z(x)]^2}\partial_E c - \frac{z(x)}{1 - z(x)}\partial_E p + \frac{1}{1 - z(x)}\partial_E q}. \tag{20}$$

Henceforth, we assume that the derivatives of the coefficients p, q and c with respect of E in (20) are constant, which requires that the coefficients are linear with respect to E [15–17]. In terms of these settings, the coefficients become

$$c(E) = -c_0 E + a_c, \qquad p(E) = -p_0 E + a_p, \qquad q(E) = -q_0 E + a_q, \tag{21}$$

where c_0, p_0, q_0, a_c, a_p and a_q are six real parameters. A straightforward algebraic manipulation permits to express the generating function $S(x)$ through the differential equation in $z(x)$

$$S(x) = \frac{4z^2(x)[1 - z(x)]^2}{R[z(x)]}, \tag{22}$$

where

$$R[z(x)] = p_0 z^2(x) + (4c_0 - p_0 - q_0)z(x) + q_0. \tag{23}$$

Substituting now (19) and (21) into (17) we obtain

$$V_{eff}(x) = \frac{a_p z^2 - (a_p + a_q - 4a_c + 1)z + a_q + 1}{R[z(x)]}$$

$$+ \frac{5}{32m}\frac{S'^2}{S^2} - \frac{1}{8m}\frac{S''}{S} + \frac{m'}{16m^2}\frac{S'}{S} + U_m^{(\eta,\varepsilon)}(x), \tag{24}$$

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[1]This replacement was introduced with the idea to ensure that $\xi(x)$ is real.

where

$$U_m^{(\eta,\varepsilon)}(x) = \left[\frac{4(1+2\eta)^2 + 16\varepsilon(1+\eta) + 5}{32}\right]\frac{m'^2(x)}{m^3(x)} - \frac{2\varepsilon+1}{8}\frac{m''(x)}{m^2(x)}. \tag{25}$$

Knowing (22), $S'(x)$ and $S''(x)$ can be recast in terms of $z(x)$ leading, after long and straightforward algebras, to express the effective potential (24) in the form

$$V_{eff}(x) = \frac{a_p z^2 - (a_p + a_q - 4a_c + 1)z + a_q + 2}{R[z(x)]}$$

$$+ \left[p_0 + \frac{(4c_0 - q_0)(2z-1) + p_0}{z(z-1)} - \frac{5}{4}\frac{\Delta}{R[z(x)]}\right]\left[\frac{z(z-1)}{R[z(x)]}\right]^2 \tag{26}$$

where $V(x) = V_{eff}(x) - U_m^{(\eta,\varepsilon)}(x)$ and $\Delta = (4c_0 - p_0 - q_0)^2 - 4p_0 q_0$.

We recognize in (26) the general Natanzon hypergeometric potentials [12, 17, 26, 27]. The corresponding bound-state spectra can be determined from (18), taking into account (3), following the identity

$$\sqrt{q+2} - \sqrt{p+1} - \sqrt{4c+1} \equiv 2n+1. \tag{27}$$

3 A Particular Example: Ginocchio Potentials

Probably the most-known member of the Natanzon-class is the Ginocchio potentials [28–30] which have as an important feature the possibility to be reduced to the Pöschl-Teller potential [31] in the one-dimension case and to the Generalized Pöschl-Teller potential in the radial case. It is a perfect example of "implicit" potentials; i.e. it is expressed in terms of a function $z(x)$ which is known only in the implicit form $x(z)$. Consequently, the bound-state spectra are then given by a more complicated form. Setting the appropriate-parameter choices

$$c_0 = \frac{1}{4\gamma^4}, \quad a_c = -\frac{1}{4}, \quad p_0 = \frac{1-\gamma^2}{\gamma^4},$$

$$a_p = \left(j + \frac{1}{2}\right)^2 - 1, \quad q_0 = 0, \quad \text{and} \quad a_q = -\frac{7}{4},$$

and combining (22) to (19), we obtain the dimensionless-mass integral

$$\mu(x) = \int^x dy \sqrt{m(y)} = \frac{1}{2\gamma^2}\int^{z(x)} \frac{ds(x)}{1 - s(x)}\sqrt{1 - \gamma^2 + \frac{\gamma^2}{s(x)}}. \tag{28}$$

By defining a new variable transformation $s(x) = \tanh^2 u(x)$, (28) is reduced to

$$\mu(x) = \frac{1}{\gamma^2}\int^{z(x)} du(x)\frac{\sqrt{\gamma^2 + \sinh^2 u(x)}}{\cosh u(x)}, \tag{29}$$

where it is impossible to get $z[\mu](x)$, the solution of (29) in closed form; rather only an implicit $\mu[z](x)$ function can be determined given by

$$\mu(x) = \frac{1}{\gamma^2}\tanh^{-1}\left[\frac{\sinh z(x)}{\sqrt{\gamma^2 + \sinh^2 z(x)}}\right] + \frac{\sqrt{\gamma^2-1}}{\gamma^2}\tan^{-1}\left[\frac{\sqrt{\gamma^2-1}\sinh z(x)}{\sqrt{\gamma^2 + \sinh^2 z(x)}}\right]. \tag{30}$$

By inserting the parameters mentioned above into (26), taking into consideration (23), we end up obtaining the expression of the Ginocchio potentials either in its "hyperbolic form" [30], given by

$$V_{hyp}(x) = -\gamma^4 \frac{j(j+1) - \gamma^2 + 1}{\gamma^2 + \sinh^2 z(x)} - \frac{3}{4} \frac{\gamma^4 (3\gamma^2 - 1)(\gamma^2 - 1)}{[\gamma^2 + \sinh^2 z(x)]^2} + \frac{5}{4} \frac{\gamma^6 (\gamma^2 - 1)^2}{[\gamma^2 + \sinh^2 z(x)]^3},$$
(31)

or in its "polynomial form" [22, 23, 28, 29]

$$V_{pol}(x) = \left[-\gamma^2 j(j+1) + \frac{1 - \gamma^2}{4} \{2 - (7 - \gamma^2) y^2(x) + 5(1 - \gamma^2) y^4(x)\} \right] (1 - y^2(x)), \quad (32)$$

once the variable transformation

$$y(x) = \frac{\sinh z(x)}{\sqrt{\gamma^2 + \sinh^2 z(x)}},$$
(33)

is introduced in (31), where $-1 \leq y \leq 1$.

Proceeding now to squaring (27), then it is easy to obtain the expression of the bound-state spectra [22, 23, 28–30]

$$E_n = -\left[\sqrt{(1 - \gamma^2)\left(2n + \frac{1}{2}\right)^2 + \gamma^2 \left(j + \frac{1}{2}\right)^2} - \left(2n + \frac{1}{2}\right) \right]^2, \quad (34)$$

where $n = 0, 1, 2, \ldots, [j]$.[2]

4 Conclusion

The conformal mappings have been used to generate the general Natanzon hypergeometric potentials (GNHP) endowed with a position-depend mass in the framework of the $\mathfrak{su}(1, 1)$ group representation, and as an example, we have derived the Ginocchio potentials and corresponding bound-state spectra as well. Here the particular interest carried upon the conformal mappings is due essentially to the fact that the variable z, as defined in Natanzon-class potentials, belongs to the interval $[0, 1]$, this led us to establish the connection between $\xi \in D$ and $z \in D_*$. To be more precise, it has been shown that the GNHP can be deduced under the linear-fractional function (16) specified by a conformal mapping; this means that the function z, in (16), specifies a mapping under which the points of the real axis $\Im m \Omega = const$, where $\xi = \exp[2i\Omega]$,[3] are one-sheeted correspondence with the points of the contour $|z| \leq 1$. Consequently, this function performs a conformal mapping of the upper half-plane onto the interior of the unit circle (see Appendix).

Thus we can conclude that the conformal mappings can be considered as one of powerful methods of generating the exactly (may be also quasi-exactly) potentials from a different perspective using only the geometric aspects, aspects that will be useful in visualizing the connection between the domains.

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[2] $[j]$ means the integer part of j.

[3] For discussion of such function, cf. the formulas (A1) and (A2) in Appendix.

Appendix: Discussion of Conformal Mappings

As outlined in the introduction, our immediate purpose in this appendix is to see how the transformation given in (16) can be deduced from a conformal mappings specified by some elementary and analytic functions.

Given an analytic function $w = f(z)$ in a domain D, to each point $z \in D$ there corresponds a definite point on the complex-plan of the variable $w \in D_*$. If this correspondence between z and w is one-to-one, then the function $w = f(z)$ is called one-sheeted. In case of such correspondence, we say that there is a mapping of the domain D onto the domain D_*. The point $w \in D_*$ is called the image of the point $z \in D$ and the point z is called the original of the point w [32–34].

Let the domain D belongs to the z-plan. The introduction of the transformation

$$\mathfrak{z} = 2iz, \tag{A1}$$

where $\mathfrak{z} = x_1 + iy_1$, allows to perform a pure rotation through an angle $\frac{\pi}{2}$ and a double dilatation of our domain. Then we make up the exponential function

$$\mathcal{Z} = \exp[\mathfrak{z}]. \tag{A2}$$

From D, let us choose $\Re e\, z$ defined in the band: $-\frac{\pi}{4} \leq \Re e\, z \leq \frac{\pi}{4}$, with the correspondence of three neighbourhood points $f(\pm\frac{\pi}{4}) = \pm 1$ and $f(i\infty) = i$ (here $i\infty$ indicates the point located at infinity in the direction of the imaginary-axis of the z-plan). Consequently, the transformations (A1) and (A2) transform the band $-\frac{\pi}{2} \leq \Im m\, \mathfrak{z} \leq \frac{\pi}{2}$, on which the function (A1) transforms $\Re e\, z$, onto the upper half-plan $\Re e\, \mathcal{Z} > 0$ (in fact as $\mathcal{Z} = e^{x_1} e^{iy_1}$, then $|\mathcal{Z}| = e^{x_1}$ varies from 0 to ∞ and $\arg \mathcal{Z} = y_1$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$). It remains to transform this upper half-plan onto the unit circle so that the points $\mathcal{Z} = i, -i, 0$ which correspond, respectively, to the points $z = \frac{\pi}{4}, -\frac{\pi}{4}, i\infty$ have as images the points $w = 1, -1, i$. This problem can be solved by connecting the three initial points z to the three final points w_α, with $\alpha = 1, 2, 3$, by the means of the transformation [32]

$$\frac{w - w_1}{w - w_3} \frac{w_2 - w_3}{w_2 - w_1} = \frac{\mathcal{Z} - z_1}{\mathcal{Z} - z_3} \frac{z_2 - z_3}{z_2 - z_1}, \tag{A3}$$

leading to the linear-fractional (homographic) functions

$$w = \frac{1}{i} \frac{\mathcal{Z} - 1}{\mathcal{Z} + 1}. \tag{A4}$$

Substituting in (A4) the expressions of (A1) and (A2) we find the solution of the problem as

$$w = \frac{1}{i} \frac{\exp[2iz] - 1}{\exp[2iz] + 1} \equiv \tan z, \tag{A5}$$

where $-1 \leq \Re e\, w \leq 1$. Finally, taking into account the inverse of (A4), we have

$$\mathcal{Z} = \frac{1 + iw}{1 - iw}, \tag{A6}$$

and after performing the transformation $w \to v = \sqrt{w}$ (i.e. $\rho = r^{1/2}$ and $2\varphi = \theta$) one ends up obtaining the desired expression (16). We now have two points in the v-plan corresponding to one point in the w-plan. The important point here is that we can make the function

v a single-valued function instead of a double-valued function if we agree to restrict θ to a range such as $0 \leq \theta \leq 2\pi$ [34]. This may be done by agreeing never to cross the line $\theta = 0$ in the w-plan. Such a line of demarcation is known as a cut line, of which the mean purpose consists to restrict the argument of w leading to write $\Re e \, w \in [0, 1]$. Then $w = 0$ is a branch point, which brings to conclude that $v = \sqrt{w}$ not being analytic at the point $w = 0$.

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